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LETTER TO THE EDITOR

A characterisation of higher-order Noether symmetries

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Abstract. It is shown that higher-order Noether symmetries of Lagrangian systems, as introduced by Sarlet and Cantrijn, can be transformed to ordinary Noether symmetries by an appropriate type (1, 1) tensor field.

Let L be a Lagrangian function, defined on the evolution space manifold $\mathbb{R}\times TM$ and let θ_L be the associated Cartan 1-form. The Euler-Lagrange equations corresponding to L are governed by the second-order differential equation field Γ satisfying $i_\Gamma\,\mathrm{d}\theta_L=0$ and $\langle\Gamma,\mathrm{d}t\rangle=1$. A dynamical symmetry of Γ is a vector field Y on $\mathbb{R}\times TM$ which commutes with Γ and (without loss of generality) is further assumed to satisfy $\langle Y,\mathrm{d}t\rangle=0$. A dynamical symmetry Y of Γ is said to be a Noether symmetry of order k (Sarlet and Cantrijn 1981) if

$$\mathscr{L}_Y^k\,\mathrm{d}\theta_L=0. \tag{1}$$

Locally, (1) implies that

$$i_Y\,\mathrm{d}(\mathscr{L}_Y^{k-1}\theta)=\mathrm{d}G, \tag{2}$$

for some function G, and G turns out to be a first integral of Γ. The case of Noether symmetries corresponds to $k=1$. When a kth-order Noether symmetry happens to be of point type, it does not essentially differ from an ordinary Noether symmetry. Indeed, point symmetries always give rise to new functions L' (Marmo and Saletan 1977) and a kth-order Noether symmetry which is of point-type can be considered as a (first-order) Noether symmetry with respect to some new Lagrangian L' (though L' need no longer be regular). One may therefore wonder whether higher-order Noether symmetries may not always be somehow related to conventional Noether symmetries. We will show here that indeed they are, through a process which involves type (1, 1) tensor fields of the same nature as those discussed by Crampin (1983) and Crampin *et al* (1983).

To each dynamical symmetry Y of Γ, we associate a whole class of type (1, 1) tensor fields $R_Y^{(k)}$ (the subscript Y indicating the dependence on the given symmetry), defined by

$$i_{R_Y^{(k)}(X)}\,\mathrm{d}\theta_L=i_X(\mathscr{L}_Y^k\,\mathrm{d}\theta_L), \tag{3}$$

$$\langle R_Y^{(k)}(X),\mathrm{d}t\rangle=0, \tag{4}$$

for all vector fields X on $\mathbb{R}\times TM$. Clearly, for $k=1$ we recover the tensor field R as defined by Crampin (1983).

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Proposition 1. If X is a dynamical symmetry of Γ, then so is $R_Y^{(k)}(X)$ for all $k=1,2,\ldots$.

Proof. Taking the Lie derivative with respect to Γ of equation (3) and using well known properties of Lie derivatives and contraction of forms with vector fields, we obtain

$$i_{R_Y^{(k)}(X)}\mathscr{L}_\Gamma\,\mathrm{d}\theta_L + i_{[\Gamma,R_Y^{(k)}(X)]}\,\mathrm{d}\theta_L = i_X\mathscr{L}_\Gamma(\mathscr{L}_Y^k\,\mathrm{d}\theta_L) + i_{[\Gamma,X]}\mathscr{L}_Y^k\,\mathrm{d}\theta_L.$$

The first term on each side of this equation is zero because $i_\Gamma\,\mathrm{d}\theta_L=0$ and $\mathscr{L}_\Gamma$ commutes with $\mathscr{L}_Y$. Knowing further that $[\Gamma,X]=0$, we thus find

$$i_{[\Gamma,R_Y^{(k)}(X)]}\,\mathrm{d}\theta_L=0,$$

which together with (4) implies $[\Gamma, R_Y^{(k)}(X)]=0$.

In particular, proposition 1 tells us that $R_Y^{(k)}(Y)$ will be a symmetry for all k and we now easily obtain the following characterisation of higher-order Noether symmetries.

Proposition 2. Y is a kth-order Noether symmetry if and only if $R_Y^{(k-1)}(Y)$ is a Noether symmetry.

Proof.

$$\begin{aligned}\mathscr{L}_{R_Y^{(k-1)}(Y)}\,\mathrm{d}\theta_L &= \mathrm{d}i_{R_Y^{(k-1)}(Y)}\,\mathrm{d}\theta_L,\\ &= \mathrm{d}i_Y\mathscr{L}_Y^{k-1}\,\mathrm{d}\theta_L, \qquad \text{(from (3))}\\ &= \mathscr{L}_Y^k\,\mathrm{d}\theta_L,\end{aligned}$$

from which the result is obvious.

We complete the picture by establishing a recursion relation which could be used as an alternative definition of the tensor fields $R_Y^{(k)}$ for $k>1$.

Proposition 3. $R_Y^{(k)}=R_Y^{(1)}\circ R_Y^{(k-1)}+\mathscr{L}_Y R_Y^{(k-1)}$.

Proof. For an arbitrary vector field X, we have

$$\begin{aligned}i_{R_Y^{(1)}\circ R_Y^{(k-1)}(X)}\,\mathrm{d}\theta_L &= i_{R_Y^{(k-1)}(X)}(\mathscr{L}_Y\,\mathrm{d}\theta_L),\\ &= \mathscr{L}_Y i_{R_Y^{(k-1)}(X)}\,\mathrm{d}\theta_L + i_{[R_Y^{(k-1)}(X),Y]}\,\mathrm{d}\theta_L,\\ &= \mathscr{L}_Y i_X(\mathscr{L}_Y^{k-1}\,\mathrm{d}\theta_L) - i_{\mathscr{L}_Y(R_Y^{(k-1)}(X))}\,\mathrm{d}\theta_L,\\ &= i_X\mathscr{L}_Y^k\,\mathrm{d}\theta_L + i_{[Y,X]}\mathscr{L}_Y^{k-1}\,\mathrm{d}\theta_L - i_{(\mathscr{L}_Y R_Y^{(k-1)})(X)}\,\mathrm{d}\theta_L - i_{R_Y^{(k-1)}([Y,X])}\,\mathrm{d}\theta_L,\\ &= i_{R_Y^{(k)}(X)}\,\mathrm{d}\theta_L - i_{(\mathscr{L}_Y R_Y^{(k-1)})(X)}\,\mathrm{d}\theta_L, \end{aligned}\tag{5}$$

where repeated use has been made of (3), for different values of k. Secondly, we have

$$\begin{aligned}\langle(R_Y^{(1)}\circ R_Y^{(k-1)}+\mathscr{L}_Y R_Y^{(k-1)})(X),\mathrm{d}t\rangle &= \langle(\mathscr{L}_Y R_Y^{(k-1)})(X),\mathrm{d}t\rangle,\\ &= \langle\mathscr{L}_Y(R_Y^{(k-1)}(X)),\mathrm{d}t\rangle - \langle R_Y^{(k-1)}([Y,X]),\mathrm{d}t\rangle,\\ &= \mathscr{L}_Y\langle R_Y^{(k-1)}(X),\mathrm{d}t\rangle - \langle R_Y^{(k-1)}(X),\mathscr{L}_Y\,\mathrm{d}t\rangle,\\ &= 0.\end{aligned}\tag{6}$$

Here, use has been made of (4) and of $\langle Y, \mathrm{d}t\rangle = 0$. The proposition now immediately follows from (5) and (6).

As a final remark, note that the proof of the property $\mathscr{L}_\Gamma R = 0$ in Crampin (1983) will also apply to the tensor fields $R_Y^{(k)}$ defined by (3) and (4), i.e. we will have

$$\mathscr{L}_\Gamma R_Y^{(k)} = 0. \tag{7}$$

Thus, all these tensor fields give rise to Lax type equations and related first integrals. It is not to be expected, however, that the $R_Y^{(k)}$ could continue to provide us with more and more independent first integrals. The recursion relations of proposition 3 indeed explain how first integrals associated with the invariance of $R_Y^{(k)}$ will, roughly speaking, be constructed out of powers of first integrals coming from $R_Y^{(1)}$ and Lie derivatives of them.

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